

Enhanced Approximation of Labeled Multi-object Density based on Correlation Analysis

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Abstract—Multi-object density is a fundamental descriptor of a point process and has ability to describe the randomness of number and values of objects, as well as the statistical correlation between objects. Due to its comprehensive nature, it usually has a complicate mathematical structure making the set integral suffering from the curse of dimension and the combinatorial nature of the problem. Hence, the approximation of multi-object density is a key research theme in point process theory or finite set statistics (FISST). Conventional approaches usually discard part or all of statistical correlation mechanically in return for computational efficiency, without considering the real situation of correlation between objects. In this paper, we propose an enhanced approximation of labeled multi-object (LMO) density which evaluates the correlation between objects adaptively and factorizes the LMO density into densities of several independent subsets according to the correlation analysis. Besides, to get a tractable factorization of LMO density, we derive the set marginal density of any subset of the universal labeled RFS, the generalized labeled multi-Bernoulli (GLMB) RFS family and its subclasses. The proposed method takes into account the simplification of the complicate structure of LMO density and the reservation of necessary correlation at the same time.

I. INTRODUCTION

In multi-object inference, the mission is to simultaneously estimate the number of objects as well as their individual states. The applications of multi-object inference involves a wide range of areas, such as forestry [1], biology [2], physics [3], computer vision [4], wireless networks [5], communications [6], multi-target tracking [7], [8], and robotic [9]. The states of objects in multi-object systems such as the coordinates of molecules in a liquid/crystal, trees in a forest, stars in a galaxy is a typical point pattern which is modeled by point processes (specifically simple finite point processes or random finite sets (RFS)) derived from stochastic geometry. The point process theory provides the tools for characterizing the underlying laws of the point patterns. Moreover, finite set statistics (FISST) [10] proposed by Mahler also provides mathematical tools for dealing with RFSs based on a notion of integration and density that is consistent with point process theory.

A fundamental descriptor of point processes is multi-object probability density which can capture the uncertainty of the number and values of objects, as well as the statistical correlation between objects. Due to its comprehensive nature, the multi-object density usually has a complicate mathematical structure, namely, the multiple hypotheses involving different cardinalities, and the high-dimensional densities conditional

on given cardinalities. The core of multi-object estimation is dynamic Bayesian inference. Computing the posterior via Bayes rule requires integrating the product of the prior and likelihood function to obtain the normalizing constant. This integration poses significant practical challenges especially for RFS prior because the complicate structure of multi-object probability density makes the set integral suffer from the curse of dimensionality and the inherently combinatorial nature of the problem.

To solve these problems, tractable approximations of multi-object probability density are necessary and two points during the approximation should be remarked. Firstly, statistical independence between objects can be utilized to enable the parallel implementation to reduce both the number of combination and the dimension of joint density. Secondly, statistical correlation between objects also should be emphasized when required. Statistical correlation usually comes from the ambiguous observation (relative to multiple objects) when considering the posterior multi-object density, or from the interactions between objects in Markov point processes [11]. If one does not consider the correlation when objects strongly depend on each other, multi-object estimation may have a big deviation.

Conventional approach usually approximates multi-object probability density as a certain class of density. There exist two categories of approximate densities: one is to completely discard the correlation between objects and assume thorough independence of objects, such as Poisson process [10], [12], independent identically distributed (i.i.d) process [10], [13], multi-Bernoulli (MB) density [10], [14], [15]. While this kind of densities enjoy many analytical properties, it has been shown that sometimes they are too simplistic for the dynamic Bayesian inference of point processes in complicated scenarios [15]. The other is to cast away only a part of correlation between objects with the typical examples generalized labeled multi-Bernoulli (GLMB) RFS family [16]–[19] and its subclasses. The advantages of the class of GLMB density is that it is a conjugate prior that is also closed under the Chapman-Kolmogorov equation for the standard multi-object system model. Moreover, the set integral of GLMB density only involves the integrals on single-object space getting rid of the curse of dimensionality. However, the class of GLMB densities are not necessarily closed under generic multi-object system because it still assumes independence of objects under each hypothesis involving the existence of different objects. To summarize, the conventional approximate approaches usually

discard part or all of statistical correlation mechanically in return for computational efficiency, without regard for the real situation of correlation between objects.

With the recent development of labeled set filters and their advantageous performance compared to previous (unlabeled) random set filters, such as identifying object identities, simplifying the multi-object transition kernel, the study on point processes turns to the labeled set gradually. In this paper, we propose an enhanced approximate approach for labeled multi-object (LMO) density [21] which does approximation based the results of correlation analysis. The proposed method will not approximate the LMO density using a certain type of distribution mechanically regardless the real correlation between objects, but evaluate the correlation between objects adaptively and factorize the LMO density into densities of several independent subsets according to correlation analysis. The proposed method takes into account the simplification of the complicate structure of LMO density and the reservation of necessary correlation at the same time. Note that the proposed method also can be used to make the further approximation of small class of LMO density, such as GLMB RFS family.

The key point of the proposed approximation is the set marginal density. However, the computation of set marginal density is not mature. In [22], we preliminarily give the concept of set marginal and its computing method for joint multi-Bernoulli RFS. In this paper, we further derive the analytical expressions of set marginal density for the universal LMO density, GLMB density family and some subclasses of GLMB density including δ -GLMB and Marginal δ -GLMB (M δ -GLMB) density, which guarantee the proposed approximation has great practicability.

II. BACKGROUND

A. Notations

We adhere to the convention that single-object states are represented by lowercase letters, e.g., \mathbf{x} , while multi-object states are represented by uppercase letters, e.g., \mathbf{X} , X . To distinguish labeled states and distributions from the unlabeled ones, bold-type letters are adopted for the labeled ones, e.g., \mathbf{x} , \mathbf{X} , $\boldsymbol{\pi}$. Moreover, blackboard bold letters represent spaces, e.g., the state space is represented by \mathbb{X} , the label space by \mathbb{L} . The collection of all finite sets of \mathbb{X} is denoted by $\mathcal{F}(\mathbb{X})$.

B. Labeled RFS and LMO Density

A *labeled RFS* is an RFS whose elements are identified by distinct labels [16], [17]. A *labeled RFS* with (kinematic) state space \mathbb{X} and (discrete) label space \mathbb{L} is an RFS on $\mathbb{X} \times \mathbb{L}$ such that each realization \mathbf{X} has distinct labels, i.e., $|\mathcal{L}(\mathbf{X})| = |\mathbf{X}|$. A labeled RFS and its unlabeled version have the same cardinality distribution. For an arbitrary labeled RFS, its multi-object density can be represented as the expression given in Lemma 1 [21], and our main results in this paper follow from this expression.

Lemma 1. *Given an LMO density $\boldsymbol{\pi}$ on $\mathcal{F}(\mathbb{X} \times \mathbb{L})$, and for any positive integer n , we define the joint existence probability*

of the label set $\{\ell_1, \ell_2, \dots, \ell_n\}$ by

$$\omega(\{\ell_1, \dots, \ell_n\}) = \int \boldsymbol{\pi}(\{(x_1, \ell_1), \dots, (x_n, \ell_n)\}) d(x_1, \dots, x_n) \quad (1)$$

and the joint probability density on \mathbb{X}^n of the states x_1, \dots, x_n conditional on their corresponding labels ℓ_1, \dots, ℓ_n by

$$p(\{(x_1, \ell_1), \dots, (x_n, \ell_n)\}) = \frac{\boldsymbol{\pi}(\{(x_1, \ell_1), \dots, (x_n, \ell_n)\})}{\omega(\{\ell_1, \dots, \ell_n\})} \quad (2)$$

Thus, the LMO density can be expressed as

$$\boldsymbol{\pi}(\mathbf{X}) = \omega(\mathcal{L}(\mathbf{X}))p(\mathbf{X}). \quad (3)$$

C. GLMB RFS Family and Its Subclasses

GLMB RFS family [16] is a class of tractable labeled RFS whose density is conjugate with standard multi-object likelihood function, and is closed under the multi-object Chapman-Kolmogorov equation with respect to the standard multi-object motion model.

A GLMB RFS is a labeled RFS with state space \mathbb{X} and (discrete) label space \mathbb{L} distributed according to

$$\boldsymbol{\pi}_{GLMB}(\mathbf{X}) = \Delta(\mathbf{X}) \sum_{c \in \mathbb{C}} w^{(c)}(\mathcal{L}(\mathbf{X})) [p^{(c)}]^{\mathbf{X}} \quad (4)$$

where \mathbb{C} is a discrete index set, $w^{(c)}(L)$ and $p^{(c)}$ satisfy

$$\begin{aligned} \sum_{L \subseteq \mathbb{L}} \sum_{c \in \mathbb{C}} w^{(c)}(L) &= 1 \\ \int p^{(c)}(x, \ell) dx &= 1. \end{aligned} \quad (5)$$

A δ -GLMB RFS with state space \mathbb{X} and discrete label space \mathbb{L} is a special case of GLMB RFS with

$$\begin{aligned} \mathbb{C} &= \mathcal{F}(\mathbb{L}) \times \Xi \\ \omega^{(c)}(\mathcal{L}) &= \omega^{(I, \xi)}(L) = \omega^{(I, \xi)} \delta_I(L) \\ p^{(c)} &= p^{(I, \xi)} = p^{(\xi)} \end{aligned} \quad (6)$$

where Ξ is a discrete space, i.e., it is distributed according to

$$\boldsymbol{\pi}_{\delta GLMB}(\mathbf{X}) = \Delta(\mathbf{X}) \sum_{(I, \xi) \in \mathcal{F}(\mathbb{L}) \times \Xi} \omega^{(I, \xi)} \delta_I(\mathcal{L}(\mathbf{X})) [p^{(\xi)}]^{\mathbf{X}}. \quad (7)$$

An marginalized δ -GLMB density $\boldsymbol{\pi}_{M\delta GLMB}$ corresponding to the δ -GLMB density $\boldsymbol{\pi}_{\delta GLMB}$ in (7) is a probability density of the form

$$\boldsymbol{\pi}_{M\delta GLMB}(\mathbf{X}) = \Delta(\mathbf{X}) \sum_{I \in \mathcal{F}(\mathbb{L})} \omega^{(I)} \delta_I(\mathcal{L}(\mathbf{X})) [p^{(I)}]^{\mathbf{X}} \quad (8)$$

where

$$\begin{aligned} \omega^{(I)} &= \sum_{\xi \in \Xi} \omega^{(I, \xi)} \\ p^{(I)}(\mathbf{x}, \ell) &= 1_I(\ell) \frac{1}{\omega^{(I)}} \sum_{\xi \in \Xi} \omega^{(I, \xi)} p^{(\xi)}(\mathbf{x}, \ell). \end{aligned} \quad (9)$$

D. δ -GLMB Density Approximation of LMO Density

An arbitrary LMO density can be approximated as a tractable δ -GLMB density, which is applied to δ -GLMB filter for generic observation model [21].

Lemma 2. *Given any LMO density π , the δ -GLMB density which preserves the cardinality distribution and probability hypothesis density (PHD) of π , and minimizes the Kullback-Leibler divergence from π , is given by*

$$\hat{\pi}_{\delta\text{GLMB}}(\mathbf{X}) = \Delta(\mathbf{X}) \sum_{I \in \mathcal{F}(\mathbb{L})} \hat{\omega}^{(I)} \delta_I(\mathcal{L}(\mathbf{X})) [\hat{p}^{(I)}]^{\mathbf{X}} \quad (10)$$

where

$$\begin{aligned} \hat{\omega}^{(I)} &= \omega(I) \\ \hat{p}^{(I)}(x, \ell) &= 1_I(\ell) p_{I-\{\ell\}}(x, \ell) \\ p_{\{\ell_1, \dots, \ell_n\}}(x, \ell) &= \\ &\int p(\{(x, \ell), (x_1, \ell_1), \dots, (x_n, \ell_n)\}) d(x_1, \dots, x_n). \end{aligned} \quad (11)$$

E. Correlation Coefficient

The correlation coefficient [23] is a measure that determines the degree to which two random variables are correlated. The most commonly used measure is the Pearson's correlation coefficient, denoted by ρ , commonly called simply "the correlation coefficient", which is sensitive only to a linear relationship between two variables. The correlation coefficient between two random variables A and B with $\text{cov}(A, B)$ the covariance of A and B , and σ_A, σ_B the standard deviations, is defined as:

$$\rho_{A,B} = \frac{\text{cov}(A, B)}{\sigma_A \sigma_B}. \quad (12)$$

Our results in this paper follow from this correlation coefficient.

F. Kullback-Leibler divergence

In probability theory and information theory, the Kullback-Leibler divergence (KLD) [34] is a measure of the difference between two probability distributions, and its extension to multi-object densities $f(X)$ and $g(X)$ is given in [35] by

$$D_{KL}(f; g) = \int f(X) \log \frac{f(X)}{g(X)} \delta X \quad (13)$$

where the integral in (13) is a set integral.

III. ENHANCED APPROXIMATE STRATEGY BASED ON CORRELATION ANALYSIS

The LMO density usually is approximated as a given type of multi-object density. In [21], it proposed a tractable δ -GLMB density approximation for an arbitrary LMO density, which matches the PHD and cardinality distribution of LMO density. For standard multi-object system, GLMB density is a closed solution [16], and it can be further approximated using LMB density [20] or M δ -GLMB density [19]. These approximations usually discard all or part of correlation of original LMO density mechanically for the sake of computation efficiency,

without respect for the real situation of correlation between objects.

Actually, correlation of objects play a important role in multi-object estimation. On one hand, when objects exhibit none correlation, the statistical independence can be utilized to enable parallel implementation, and thus simplify computation and enhance estimation performance [19], [24]–[30]. On the other hand, when objects are strongly correlation with each other, their statistics should be jointly considered, or it will produce poor estimation like the the aforementioned strategies.

In practice, the real situation of correlation between objects is usually complicate. Empirical data suggests that in most scenarios not all objects have correlation with each other, but only a small faction of objects has correlation with the other small faction of objects and which object has correlation with which is usually unknown and time-varying.

In this section, we present an enhanced approximate strategy for the approximation of LMO density, in which the correlation between different objects is estimated adaptively, and the original LMO density is factorized into densities of several independent subsets according to the correlation estimate. Besides, we derive the analytical expression of set marginal density of any subsets of the universal LMO RFS and GLMB RFS family.

A. Correlation Estimate

Firstly, we introduce a concept of *basic component* of labeled RFS in Definition 1, which is important in estimating the correlation between objects.

Definition 1. *For an arbitrary labeled RFS Ψ on space $\mathbb{X} \times \mathbb{L}$ (\mathbb{L} is a finite label space), it can be seen as the union of $|\mathbb{L}|$ random subsets, i.e., $\Psi = \biguplus_{\ell \in \mathbb{L}} \psi_\ell$, with each ψ_ℓ on space $\mathbb{X} \times \{\ell\}$. We refer each random finite subset $\psi_\ell, \ell \in \mathbb{L}$ to as a basic component (BC) of Ψ .*

A BC ψ_ℓ , namely, the random finite subset related to the target with label ℓ , is a labeled Bernoulli RFS which is either the empty set or the singleton set $\{\mathbf{x}, \ell\}$.

A BC ψ_ℓ is the mathematical representation of the object ℓ , and can describe both the uncertainty of existence and the randomness of state of object ℓ . Hence, to evaluate the correlation between different BCs, we should consider comprehensively from two aspects: 1) the correlation of objects' existences ; 2) the correlation of object states.

1) *Absolute Correlation Coefficient of Existence, $\alpha_{\ell, \ell'}$* : To describe its uncertainty of existence, we define a random variable E_ℓ for each BC ψ_ℓ as

$$E_\ell = \begin{cases} 0, & \psi_\ell = \emptyset \\ 1, & \psi_\ell = \{\mathbf{x}, \ell\} \end{cases}, \quad \ell \in \mathbb{L} \quad (14)$$

and the statistics of all E_ℓ s, $\ell \in \mathbb{L}$ are distributed according to the joint probability distribution

$$\Pr((\cap_{\ell \in I} \{E_\ell = 1\}) \cap (\cap_{\ell' \in \mathbb{L}/I} \{E_{\ell'} = 0\})) = \omega(I), \quad I \subseteq \mathbb{L} \quad (15)$$

where "/" denotes the different set, and $\omega(I)$ is the joint existence probability of the label set I given in Lemma 1.

We define the *absolute correlation coefficient of existence* between ψ_ℓ and $\psi_{\ell'}$, $\ell \neq \ell' \in \mathbb{L}$ as

$$\alpha_{\ell, \ell'} = |\rho_{E_\ell, E_{\ell'}}| \quad (16)$$

where $|\cdot|$ denotes the absolute value of \cdot , and $\rho_{E_\ell, E_{\ell'}}$ is the correlation coefficient between E_ℓ and $E_{\ell'}$ which can be computed from the joint existence distribution in (15) according to (12).

2) *Absolute Correlation Coefficient of State*, $\beta_{\ell, \ell'}$: To estimate correlation between states of ψ_ℓ and $\psi_{\ell'}$ is on the condition that both ψ_ℓ and $\psi_{\ell'}$ exist. Under each hypothesis (involving the existing objects with target label set $I \in \mathcal{F}(\mathbb{L})$) where I includes ℓ and ℓ' , we can compute a correlation coefficient between (\mathbf{x}, ℓ) and (\mathbf{x}', ℓ') , denoted as $\rho_{\ell, \ell'|I}$, from the corresponding conditional joint probability density $p(\mathbf{X})|(\mathcal{L}(\mathbf{X}) = I)$ defined in Lemma 1, according to (12).

For any two $\ell \neq \ell' \in \mathbb{L}$, we define the *absolute correlation coefficient of state* between ψ_ℓ and $\psi_{\ell'}$ as

$$\beta_{\ell, \ell'} = \frac{\sum_{I \in \mathcal{F}(\mathbb{L})} 1_I(\{\ell, \ell'\}) \omega(I) |\rho_{\ell, \ell'|I}|}{\sum_{I \in \mathcal{F}(\mathbb{L})} 1_I(\{\ell, \ell'\}) \omega(I)} \quad (17)$$

where $1_I(I')$ denotes an indicator function

$$1_I(I') = \begin{cases} 1, & I' \subseteq I \\ 0, & \text{otherwise} \end{cases} \quad (18)$$

$\beta_{\ell, \ell'}$ actually is the weighted sum of absolute $\rho_{\ell, \ell'|I}$ s over all hypotheses where the existing target label set I includes ℓ and ℓ' .

Definition 2. For an arbitrary labeled RFS Ψ on space $\mathbb{X} \times \mathbb{L}$ (\mathbb{L} is a finite label space), the *absolute correlation coefficient* between any two BCs ψ_{ℓ_1} and ψ_{ℓ_2} , $\ell_1 \neq \ell_2 \in \mathbb{L}$, is defined as

$$\gamma_{\ell_1, \ell_2} = \omega_E \alpha_{\ell_1, \ell_2} + \omega_S \beta_{\ell_1, \ell_2} \quad (19)$$

where $\omega_E + \omega_S = 1$ with ω_E, ω_S the weighting coefficients of $\alpha_{\ell_1, \ell_2}, \beta_{\ell_1, \ell_2}$ respectively, and $\alpha_{\ell_1, \ell_2}, \beta_{\ell_1, \ell_2}$ are the absolute correlation coefficients of existence and state defined in (16) and (17), respectively.

Note that the absolute correlation coefficient whose value goes from 0 to 1 is an indicator to evaluate the correlation between BCs comprehensively. The value is bigger, the correlation is stronger and vice versa. The value of ω_E or ω_S varies with different applications. If the correlation of state is emphasized, the value of ω_S is larger; otherwise, the value of ω_E is larger.

B. Factorization of LMO density

After estimating the correlation between different BCs, we can divide all ψ_ℓ s, $\ell \in \mathbb{L}$ into several groups such that BCs within a group have correlation, and BCs between different groups exhibit none correlation. We represent each group as the union of the BCs within the group, then Ψ can be divided into several independent random finite subsets, i.e., $\Psi = \uplus_{i=1}^N \Psi_i$.

Lemma 3. Let $\Psi = \Psi_1 \cup \dots \cup \Psi_N$ where Ψ_1, \dots, Ψ_N are statistically independent random finite subsets. The probability density of Ψ is related to the probability densities of Ψ_1, \dots, Ψ_N as follows:

$$\pi_\Psi(X) = \sum_{W_1 \uplus \dots \uplus W_n = X} \pi_{\Psi_1}(W_1) \cdots \pi_{\Psi_n}(W_n). \quad (20)$$

Lemma 3 is a conclusion provided in [10]. For the labeled RFS, we can obtain the similar conclusions as Proposition 1.

Proposition 1. If a labeled RFS Ψ on space $\mathbb{X} \times \mathbb{L}$ can be divided into N independent label random subsets Ψ_i on space $\mathbb{X} \times \mathbb{L}_i, i = 1, \dots, N$, i.e., $\Psi = \bigcup_{i=1}^N \Psi_i$ with $\mathbb{L} = \mathbb{L}_1 \uplus \dots \uplus \mathbb{L}_N$, then $f_\Psi(\mathbf{X})$ is related to the probability densities of Ψ_1, \dots, Ψ_N as follows

$$\pi_\Psi(\mathbf{X}) = \pi_{\Psi_1}(\mathbf{X} \cap \mathbb{X} \times \mathbb{L}_1) \cdots \pi_{\Psi_N}(\mathbf{X} \cap \mathbb{X} \times \mathbb{L}_N). \quad (21)$$

According to Proposition 1, we can give an approximation of LMO density which can decrease the dimension of states and reduce the number of hypotheses by utilizing independence and retain correlation if required by adopting LMO densities of independent random finite subsets. However, we still have a problem that how to compute the LMO density of each random subset from the global LMO density, which will be discussed in the following subsection.

C. Set Marginal Density

In [22], we have given concept of the set marginal density as shown in Definition 3 and its generalized computing method as shown in Lemma 4. In this section, Propositions 2–5 provide the specified method to compute the set marginal density of the universal LMO density, GLMB density and some special cases of GLMB density including δ -GLMB and $M\delta$ -GLMB densities, respectively.

Definition 3. Let Ψ be an RFS. Then for any random finite subset of Ψ , denoted by Ψ_1 , its multi-object density $f_{\Psi_1}(X)$, is called as set marginal density of Ψ_1 with respect to Ψ .

Lemma 4. Let Ψ be an RFS. Then for any random finite subset of Ψ , denoted by Ψ_1 , its set marginal density of Ψ_1 with respect to Ψ , denoted by $f_{\Psi_1}(X)$ can be derived by

$$f_{\Psi_1}(X) = \left. \frac{\delta \Pr(\Psi_1 \subseteq S, \Psi / \Psi_1 \subseteq \mathbb{X})}{\delta X} \right|_{S=\emptyset} \quad (22)$$

where “ $\delta/\delta X$ ” denotes a set derivative.

Proposition 2. Assume a labeled RFS Ψ , whose state space is $\mathbb{X} \times \mathbb{L}$ and multi-object density is $\pi_\Psi(\mathbf{X}) = \omega(\mathcal{L}(\mathbf{X}))p(\mathbf{X})$. If Ψ_1 is a random finite subset of Ψ and its state space is $\mathbb{X} \times \mathbb{L}_1$ with $\mathbb{L}_1 \subseteq \mathbb{L}$, then the set marginal density of Ψ_1 is

$$\pi_{\Psi_1}(\mathbf{X}) = \sum_{I \in \mathcal{F}(\mathbb{L}/\mathbb{L}_1)} \omega(\mathcal{L}(\mathbf{X}) \cup I) p_I(\mathbf{X}) \quad (23)$$

where

$$p_{\{\ell_1, \dots, \ell_n\}}(\mathbf{X}) = \int p(\mathbf{X} \cup \{(x_1, \ell_1), \dots, (x_n, \ell_n)\}) dx_1 \cdots dx_n. \quad (24)$$

Proposition 3. Assume a GLMB RFS Ψ , whose state space is $\mathbb{X} \times \mathbb{L}$ and multi-object density has the form of (4). If Ψ_1 is a random finite subset of Ψ and its state space is $\mathbb{X} \times \mathbb{L}_1$ with $\mathbb{L}_1 \subseteq \mathbb{L}$, then the set marginal density of Ψ_1 is

$$\pi_{\Psi_1}(\mathbf{X}) = \sum_{I \in \mathcal{F}(\mathbb{L}/\mathbb{L}_1)} \sum_{c \in \mathbb{C}} \omega^{(c)}(\mathcal{L}(\mathbf{X}) \cup I) [p^{(c)}]^{\mathbf{X}}. \quad (25)$$

Proposition 4. Assume a δ -GLMB RFS Ψ on state space is $\mathbb{X} \times \mathbb{L}$ and its multi-object density has the same form of (7). If Ψ_1 is a random finite subset of Ψ and its state space is $\mathbb{X} \times \mathbb{L}_1$ with $\mathbb{L}_1 \subseteq \mathbb{L}$, then the set marginal density of Ψ_1 is

$$\pi_{\Psi_1}(\mathbf{X}) = \sum_{I_2 \in \mathcal{F}(\mathbb{L}/\mathbb{L}_1)} \sum_{(I_1, \xi) \in \mathcal{F}(\mathbb{L}_1) \times \Xi} \delta_{I_1}(\mathcal{L}(\mathbf{X})) \omega^{(I_1 \cup I_2, \xi)} [p^{(\xi)}]^{\mathbf{X}}. \quad (26)$$

Proposition 5. Assume a $M\delta$ -GLMB RFS Ψ on state space is $\mathbb{X} \times \mathbb{L}$ and its multi-object density has the same form of (8). If Ψ_1 is a random finite subset of Ψ and its state space is $\mathbb{X} \times \mathbb{L}_1$ with $\mathbb{L}_1 \subseteq \mathbb{L}$, then the set marginal density of Ψ_1 is

$$\pi_{\Psi_1}(\mathbf{X}) = \sum_{I_2 \in \mathcal{F}(\mathbb{L}/\mathbb{L}_1)} \sum_{I_1 \in \mathcal{F}(\mathbb{L}_1)} \delta_{I_1}(\mathcal{L}(\mathbf{X})) \omega^{(I_1 \cup I_2)} [p^{(I_1)}]^{\mathbf{X}}. \quad (27)$$

IV. NUMERICAL RESULTS

Consider a labeled RFS Ψ on space $\mathbb{X} \times \mathbb{L}$, where $\mathbb{X} = \mathbb{R}$ is the field of real number and $\mathbb{L} = \{1, 2, 3\}$. We design an LMO density of Ψ shown as

$$\pi(\mathbf{X}) = \begin{cases} 0.01, & \mathbf{X} = \emptyset \\ 0.01\mathcal{N}(x; m_1, R_1), & \mathbf{X} = \{(x, 1)\} \\ 0.01\mathcal{N}(x; m_2, R_2), & \mathbf{X} = \{(x, 2)\} \\ 0.09\mathcal{N}(x; m_3, R_3), & \mathbf{X} = \{(x, 3)\} \\ 0.07\mathcal{N}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \mathbf{m}_{12}, \mathbf{R}_{12}\right), & \mathbf{X} = \{(x_1, 1), (x_2, 2)\} \\ 0.09\mathcal{N}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \mathbf{m}_{13}, \mathbf{R}_{13}\right), & \mathbf{X} = \{(x_1, 1), (x_2, 3)\} \\ 0.09\mathcal{N}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \mathbf{m}_{23}, \mathbf{R}_{23}\right), & \mathbf{X} = \{(x_1, 2), (x_2, 3)\} \\ 0.63\mathcal{N}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \mathbf{m}_{123}, \mathbf{R}_{123}\right), & \mathbf{X} = \{(x_1, 1), (x_2, 2), (x_3, 3)\} \end{cases} \quad (28)$$

where

$$\begin{aligned} m_1 &= 1, R_1 = 1 \\ m_2 &= 2, R_2 = 2 \\ m_3 &= 8, R_3 = 3 \\ \mathbf{m}_{12} &= \begin{pmatrix} 1.1 \\ 1.2 \end{pmatrix}, \mathbf{R}_{12} = \begin{bmatrix} 1.2 & 1 \\ 1 & 2.2 \end{bmatrix} \\ \mathbf{m}_{13} &= \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \mathbf{R}_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \\ \mathbf{m}_{23} &= \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \mathbf{R}_{23} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ \mathbf{m}_{123} &= \begin{pmatrix} 1.1 \\ 1.2 \\ 8 \end{pmatrix}, \mathbf{R}_{123} = \begin{bmatrix} 1.2 & 1 & 0 \\ 1 & 2.2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned} \quad (29)$$

A. Approximations of LMO density

In the subsection, we give three approximations of the LMO density in (28).

- δ -GLMB density approximation, $\hat{\pi}_{\delta GLMB}$, according to Lemma 2;
- Correlation analysis (CA) based approximation, $\hat{\pi}_{CA}$, proposed in Section III;
- CA based approximation of the approximate δ -GLMB density, $\hat{\pi}_{\delta GLMB}^{CA}$, which first approximates the LMO density as a δ -GLMB density and then approximate the resulting δ -GLMB density using the CA based method.

1) δ -GLMB Density Approximation: Based on lemma 2, we approximate (28) into a δ -GLMB density shown as

$$\hat{\pi}_{\delta GLMB}(\mathbf{X}) = \begin{cases} 0.01, & \mathbf{X} = \emptyset \\ 0.01\mathcal{N}(x; m_1, R_1), & \mathbf{X} = \{(x, 1)\} \\ 0.01\mathcal{N}(x; m_2, R_2), & \mathbf{X} = \{(x, 2)\} \\ 0.09\mathcal{N}(x; m_3, R_3), & \mathbf{X} = \{(x, 3)\} \\ 0.07 \prod_{i \in \{1, 2\}} \mathcal{N}(x_i; m_{12}^i, R_{12}^i), & \mathbf{X} = \{(x_1, 1), (x_2, 2)\} \\ 0.09 \prod_{i \in \{1, 2\}} \mathcal{N}(x_i; m_{13}^i, R_{13}^i), & \mathbf{X} = \{(x_1, 1), (x_3, 3)\} \\ 0.09 \prod_{i \in \{2, 3\}} \mathcal{N}(x_i; m_{23}^i, R_{23}^i), & \mathbf{X} = \{(x_2, 2), (x_3, 3)\} \\ 0.63 \prod_{i \in \{1, 2, 3\}} \mathcal{N}(x_i; m_{123}^i, R_{123}^i), & \mathbf{X} = \{(x_1, 1), (x_2, 2), (x_3, 3)\} \end{cases} \quad (30)$$

where

$$\begin{aligned} m_{12}^i &= \mathbf{m}_{12}(i), R_{12}^i = \mathbf{R}_{12}(i, i), i = 1, 2 \\ m_{13}^i &= \mathbf{m}_{13}(i), R_{13}^i = \mathbf{R}_{13}(i, i), i = 1, 3 \\ m_{23}^i &= \mathbf{m}_{23}(i), R_{23}^i = \mathbf{R}_{23}(i, i), i = 2, 3 \\ m_{123}^i &= \mathbf{m}_{123}(i), R_{123}^i = \mathbf{R}_{123}(i, i, i), i = 1, 2, 3. \end{aligned} \quad (31)$$

2) CA based Approximation: From (28), we can extract the distribution of (E_1, E_2, E_3) as Table I,

According to (19), we can get the absolute correlation coefficients between each BCs of Ψ , i.e., ψ_1 , ψ_2 and ψ_3 as

$$\gamma_{1,2} = \frac{\alpha_{1,2} + \beta_{1,2}}{2} = \frac{0.375 + 0.616}{2} = 0.4955 \quad (32)$$

TABLE I
THE JOINT DISTRIBUTION OF (E_1, E_2, E_3)

$E_3 \backslash E_1, E_2$	00	10	01	11
0	0.01	0.01	0.01	0.07
1	0.09	0.09	0.09	0.63

$$\gamma_{2,3} = 0 \quad (33)$$

$$\gamma_{1,3} = 0 \quad (34)$$

where $\omega_E = \omega_S = \frac{1}{2}$ in (19). Hence, we can conclude that ψ_3 is independent of ψ_1 and ψ_2 , and ψ_1 and ψ_2 do have correlation. We can divide Ψ into two independent subsets, namely, $\psi_1 \cup \psi_2$ and ψ_3 .

Let $\Psi_a = \psi_1 \cup \psi_2$ and $\Psi_b = \psi_3$. According to Proposition 2, we can compute the set marginal density of Ψ_a and Ψ_b as

$$\pi_{\Psi_a} = \begin{cases} 0.1, & \mathbf{X} = \emptyset \\ 0.1\mathcal{N}(x; m_{a,1}, R_{a,1}), & \mathbf{X} = \{(x, 1)\} \\ 0.1\mathcal{N}(x; m_{a,2}, R_{a,2}), & \mathbf{X} = \{(x, 2)\} \\ 0.7\mathcal{N}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \mathbf{m}_{a,12}, \mathbf{R}_{a,12}\right), & \mathbf{X} = \{(x_1, 1), (x_2, 2)\} \end{cases} \quad (35)$$

where

$$\begin{aligned} m_{a,1} &= 1, R_{a,1} = 1 \\ m_{a,2} &= 2, R_{a,2} = 2 \\ \mathbf{m}_{a,12} &= \begin{pmatrix} 1.1 \\ 1.2 \end{pmatrix}, \mathbf{R}_{a,12} = \begin{bmatrix} 1.2 & 1 \\ 1 & 2.2 \end{bmatrix} \end{aligned} \quad (36)$$

and

$$\pi_{\Psi_b}(\mathbf{X}) = \begin{cases} 0.1, & \mathbf{X} = \emptyset \\ 0.9\mathcal{N}(x; m_{b,3}, R_{b,3}), & \mathbf{X} = \{(x, 3)\} \end{cases} \quad (37)$$

where

$$m_{b,3} = 3, R_{b,3} = 8. \quad (38)$$

Finally, we can obtain the CA based approximation as

$$\hat{\pi}_{CA}(\mathbf{X}) = \pi_{\Psi_a}(\mathbf{X} \cap \mathbb{X} \times \mathbb{L}_a) \pi_{\Psi_b}(\mathbf{X} \cap \mathbb{X} \times \mathbb{L}_b) \quad (39)$$

with $\mathbb{L}_a = \{1, 2\}$ and $\mathbb{L}_b = \{3\}$.

3) *CA based Approximation of the Approximate GLMB Density*: Let $\hat{\Psi}$ denotes the approximate δ -GLMB RFS whose density is (30), and $\hat{\psi}_\ell$ s, $\ell \in \{1, 2, 3\}$ denote the BCs of $\hat{\Psi}$.

According to (19), we can get the absolute correlation coefficients between $\hat{\psi}_1$, $\hat{\psi}_2$ and $\hat{\psi}_3$ as

$$\gamma_{1,2} = \frac{\alpha_{1,2}}{2} = 0.1875 \quad (40)$$

$$\gamma_{2,3} = 0 \quad (41)$$

$$\gamma_{1,3} = 0 \quad (42)$$

where $\omega_E = \omega_S = \frac{1}{2}$ in (19). One can find that the approximate GLMB density does lose a part of correlation towards the original LMO density comparing (32) and (40).

Hence, we can also conclude that $\hat{\psi}_3$ is independent of both $\hat{\psi}_1$ and $\hat{\psi}_2$, and $\hat{\psi}_1$ and $\hat{\psi}_2$ do have correlation. Then we can

also divide $\hat{\Psi}$ into two independent subsets, namely, $\hat{\psi}_1 \cup \hat{\psi}_2$ and $\hat{\psi}_3$.

Let $\hat{\Psi}_a = \hat{\psi}_1 \cup \hat{\psi}_2$ and $\hat{\Psi}_b = \hat{\psi}_3$. According to Proposition 3, we can compute the set marginal density of $\hat{\Psi}_a$ and $\hat{\Psi}_b$ as

$$\pi_{\hat{\Psi}_a}(\mathbf{X}) \quad (43)$$

$$= \begin{cases} 0.1, & \mathbf{X} = \emptyset \\ 0.1\mathcal{N}(x; \hat{m}_{a,1}, \hat{R}_{a,1}), & \mathbf{X} = (x, 1) \\ 0.1\mathcal{N}(x; \hat{m}_{a,2}, \hat{R}_{a,2}), & \mathbf{X} = (x, 2) \\ 0.7 \prod_{i \in \{1,2\}} \mathcal{N}((x_i; \hat{m}_{a,12}^i, \hat{R}_{a,12}^i), & \mathbf{X} = \{(x_1, 1), (x_2, 2)\} \end{cases} \quad (44)$$

where

$$\begin{aligned} \hat{m}_{a,1} &= 1, \hat{R}_{a,1} = 1 \\ \hat{m}_{a,2} &= 2, \hat{R}_{a,2} = 2 \\ \hat{m}_{a,12}^1 &= 1.1, \hat{R}_{a,12}^1 = 1.2 \\ \hat{m}_{a,12}^2 &= 2.2, \hat{R}_{a,12}^2 = 2.2 \end{aligned} \quad (45)$$

$$\pi_{\hat{\Psi}_b}(\mathbf{X}) = \begin{cases} 0.1, & \mathbf{X} = \emptyset \\ 0.9\mathcal{N}(x; \hat{m}_{b,3}, \hat{R}_{b,3}), & \mathbf{X} = (x, 3) \end{cases} \quad (46)$$

where

$$\hat{m}_{b,3} = 3, \hat{R}_{b,3} = 8. \quad (47)$$

Hence, the CA based approximation of the approximated GLMB density is

$$\hat{\pi}_{GLMB}^{CA}(\mathbf{X}) = \pi_{\hat{\Psi}_a}(\mathbf{X} \cap \mathbb{X} \times \mathbb{L}_a) \hat{\pi}_{\hat{\Psi}_b}(\mathbf{X} \cap \mathbb{X} \times \mathbb{L}_b) \quad (48)$$

with $\mathbb{L}_a = \{1, 2\}$ and $\mathbb{L}_b = \{3\}$.

B. Computational Complexity and Approximate Error Analysis

1) *Computational Complexity*: In order to evaluate the computational complexity of the original LMO density and its approximations, some important quantities of π , $\hat{\pi}_{\delta GLMB}$, $\hat{\pi}_{CA}$ and $\hat{\pi}_{\delta GLMB}^{CA}$ are summarized as Table II,

TABLE II
COMPUTATION ANALYSIS

	T_0	T_1	T_2	T_3	Correlation loss
π	8	3	3	1	
$\hat{\pi}_{\delta GLMB}$	8	12	0	0	yes
$\hat{\pi}_{CA}$	4	3	1	0	no
$\hat{\pi}_{\delta GLMB}^{CA}$	4	5	0	0	yes (same as $\hat{\pi}_{\delta GLMB}$)

T_0 : NO. of hypotheses

T_1 : NO. of densities on \mathbb{X}

T_2 : NO. of densities on \mathbb{X}^2

T_3 : NO. of densities on \mathbb{X}^3

Note that the hypotheses here involves different existing target label sets I , and the hypothesis $I = \emptyset$ is omitted because it can be determined by other hypotheses totally.

2) *Approximate Error*: Herein, we evaluate the approximate error of three approximations in terms of KLD.

The KLD between $\hat{\pi}_{CA}$ and π , $\hat{\pi}_{\delta GLMB}$ and π , $\hat{\pi}_{\delta GLMB}^{CA}$ and π is computed as respectively, according to (13),

$$D_{KL}(\pi; \hat{\pi}_{\delta GLMB}) = 0.2404 \quad (49)$$

$$D_{KL}(\pi; \hat{\pi}_{CA}) = 0 \quad (50)$$

$$D_{KL}(\pi; \hat{\pi}_{\delta GLMB}^{CA}) = 0.2404. \quad (51)$$

We can also obtain the KLD between $\hat{\pi}_{\delta GLMB}$ and $\hat{\pi}_{\delta GLMB}^{CA}$,

$$D_{KL}(\hat{\pi}_{\delta GLMB}; \hat{\pi}_{\delta GLMB}^{CA}) = 0. \quad (52)$$

3) *Summary*: δ -GLMB approximation of LMO density actually approximates the conditional joint probability density $P(\mathbf{X})$ under each hypothesis as the product of its marginal densities and retains all hypotheses. Thus, as shown in Table II, the number of hypotheses of $\hat{\pi}_{\delta GLMB}$ are the same as π , and all the densities of $\hat{\pi}_{\delta GLMB}$ are on space \mathbb{X} . As $\hat{\pi}_{\delta GLMB}$ loses a certain degree of correlation, it has the approximate error (the KLD between π and $\hat{\pi}_{\delta GLMB}$ is not zero). Further comparing $\hat{\pi}_{\delta GLMB}^{CA}$ with $\hat{\pi}_{\delta GLMB}$, we find that $\hat{\pi}_{\delta GLMB}$ has redundant statistical information, for it can be further simplified by reducing the number of hypotheses and number of densities without approximate error.

As for $\hat{\pi}_{CA}$, it reduces the number of hypotheses by utilizing independence, and also retains high-dimensional densities to keep correlation. As shown in Table II, even though $\hat{\pi}_{CA}$ has a density on space \mathbb{X}^2 while $\hat{\pi}_{\delta GLMB}$ does not have, $\hat{\pi}_{CA}$ only has 3 densities on space \mathbb{X} while $\hat{\pi}_{\delta GLMB}$ has 12. Furthermore, the high-dimensional density of $\hat{\pi}_{CA}$ is retained in return for keeping required correlation. Hence, $\hat{\pi}_{CA}$ is a kind of approximation which can balance the computational complexity and approximate error.

V. CONCLUSION

In this paper, we proposed an enhanced approximation of labeled multi-object (LMO) density which evaluates the correlation between objects adaptively and factorizes the LMO density into densities of several independent subsets according to the correlation analysis. Furthermore, to obtain a tractable factorization of LMO density, we derived the set marginal density of any subset of the universal labeled RFS, and GLMB RFS family and its subclasses. Unlike the conventional approximate approach which sacrifices statistical correlation for computational efficiency, the proposed method takes into account the simplification of the complicate structure of LMO density and the reservation of necessary correlation at the same time.

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